SYNTHETIC UNIT HYDROGRAPHS FOR SMALL WATERSHEDS

By Don M. Gray

SYNOPSIS

To apply the unit-graph principle on an ungaged area, a representative unit graph for the watershed must be selected. A method is presented herein whereby the unit graphs for small watersheds can be synthesized from measurable topographic characteristics. In the development of the procedure, a two-parameter equation was used to describe the hydrograph. A successful linkage was established to permit the evaluation of these parameters from measurements of the length and slope of the main stream as taken from topographic maps.

INTRODUCTION

An approximation of the discharge hydrograph for a given watershed resulting from a given rainfall pattern can be obtained by application of the infiltration theory and the unit-hydrograph principle. The reliability of this approximation is limited in part by the success with which the unit hydrograph for the watershed can be derived.

Several methods can be used to develop the unit hydrograph for an ungaged area from measurable physical characteristics. In general, however, most of these methods use empirical relationship obtained either from large watersheds (10 sq miles to 10,000 sq miles) or from small watersheds (less than 1 squ mile)
Hydrologic investigations on watersheds of intermediate size are not without importance, however. A large percentage of the hydraulic structures constructed by the Soil Conservation Service and Bureau of Public Roads are designed to handle surface runoff originating from watersheds less than 25 sq miles to 50 sq miles in size. Additional hydrologic investigations on these watersheds are desirable because of the large expenditures invested annually for facilities used in the control and conservation of runoff from these areas and the relatively inadequate data on which the design of these facilities is based.

This paper describes a procedure whereby the unit hydrograph can be synthesized for small watershed areas. It presents the methodology and necessary relationships to perform this approximation once pertinent characteristics of the watershed are known.

**THEORY**

In 1932 L. K. Sherman, F. ASCE, (16)\(^2\) presented the concept of the unit graph. A unit graph is a discharge hydrograph representing 1 in. of direct runoff generated uniformly over the tributary area at a uniform rate during a specified period of time. It represents the integrated effects of all the sensibly constant physical characteristics of a watershed on the translation and storage of a given rainfall excess as it flows from the watershed. Thus, pertinent features of the unit graph would be expected to be related to the physical characteristics of the area.

Among the recent contributions to the field of hydrology has been the development of theoretical expressions to define the geometry of the instantaneous unit graph (unit graph resulting from a rainfall excess of 1 in. generated during zero time). Two mathematical expressions have been proposed; one by C. G. Edson, M. ASCE, (6), and the other by J. E. Nash (14). These are given by Eqs. 1a and 2, respectively.

**Edson.**

\[
Q_t = \frac{V}{\Gamma(z+1)} y t^z e^{-yt} \quad \ldots \ldots \ldots \ldots \quad (1a)
\]

in which \(Q_t\) denotes instantaneous discharge rate at time, \(t\); \(V\) is the volume of surface runoff; \(y\) represents the recession constant whose magnitude is greater than zero as determined from the slope of the recession curve plotted on semi-logarithmic paper; \(z\) is the exponent whose magnitude depends on the shape of the time-area concentration curve of the watershed; \(e\) is the base of the natural logarithms; and \(\Gamma\) denotes the gamma function.

By setting \(z = m^{-1}\) and simplifying, Eq. 1a can be written in a more useful form.

\[
Q_t = \frac{V y^m}{\Gamma(m)} e^{-yt} t^{m-1} \quad \ldots \ldots \ldots \ldots \quad (1b)
\]

\(^2\) Numerals in parentheses refer to corresponding items in the Appendix—Selected Bibliography.
UNIT HYDROGRAPHS

in which $Q_t$, $V$, $\Gamma$, $e$ and $t$ connote the same definition as implied previously; $n$ is the shape parameter whose magnitude depends on the storage properties of the watershed; and $k$ is the storage constant. As shown by Eqs. 1 and 2, by evaluating 2 parameters, the complete unit hydrograph can be described.

The relationships serve as a useful tool in developing a rational synthetic procedure. They provide the investigator the opportunity to seek a solution in logical sequence from reason to result. In addition, the use of point correlations as used almost exclusively in the past to relate the physical characteristics of the watershed with properties of the unit graph can be eliminated. Edson (6) suggests that the general failure encountered in correlating basin characteristics and the hydrograph properties, peak discharge and period of rise, may be attributed to the complex relation of $y$ and $z$ (Eq. 1a) so as to restrict a satisfactory tie-in. Another important advantage for describing the unit hydrograph mathematically is the adaptability of the equational form for use in high speed computers.

Data.—Hydrologic and topographic data were obtained from 42 watersheds located in Illinois, Iowa, Missouri, Nebraska, Ohio, and Wisconsin. A list of the pertinent hydrologic and topographic data from these watersheds that are used in this paper is given in Table 1. The watershed characteristics are shown in Fig. 1.

Selection of Hydrologic Data.—In the study, the unit-storm concept proposed by C.O. Wisler, F. ASCE, and E. F. Brater, F. ASCE, (22) was accepted. A unit storm is a storm of such duration that the period of surface runoff is not appreciably less for any storm of shorter duration. The duration of the unit storm varies with watershed size. For small watersheds it approaches the period of rise of the unit graph; for large watersheds, it may be only a fraction of that time.

B. S. Barnes, F. ASCE, (1), M. M. Bernard (2), and Brater (3) have suggested several criteria to be followed in selecting hydrologic data suitable for distribution-graph and/or unit-hydrograph development. These were summarized to formulate the basis of the following list of standards used in this study.

1. The rain must have fallen within the selected time unit and must not have extended beyond the period of rise of the hydrograph.
2. The storm must have been well distributed over the watershed, all stations showing an appreciable amount.
3. The storm period must have occupied a place of comparative isolation in the record.
4. The runoff following a storm must have been uninterrupted by the effects of low temperatures and unaccompanied by melting snow or ice.
5. The stage graphs or hydrographs must have a sharp, defined, rising limb culminating to a single peak and followed by an uninterrupted recession.
6. All stage graphs or hydrographs for the same watershed must show approximately the same period of rise.
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<th>Topographic Properties</th>
<th>Dimensionless Graph Properties</th>
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*a* Drainage-area size, $A = \text{plane area of the watershed in square miles which is enclosed within the topographic divide above the gaging station.}$

*b* Length of main stream, $L = \text{distance in miles along the main stream from the gaging station to the outermost point on the stream channel as defined on the topographic map (Fig. 1). The main stream is the stream of highest order that passes through the gaging station. To delineate the main stream at bifurcations, the following rules suggested by Horton (10, p. 281) were used:}

1. Starting below the junction, the main stream was projected upstream from the bifurcation in the same direction. The stream joining the main stream at the greatest angle was taken as the lower order (Fig. 1).

2. If both streams were at about the same angle to the main stream at the junction, the shorter was taken as the lower order.

*c* Slope of the main stream, $S_C = \text{slope in \% of a line drawn along the longitudinal section of the main channel in such a manner so as to have the same area enclosed by it as does the profile (Fig. 1).}$

Development of an Empirical Graph.—The hydrographs and stage graphs collected from each watershed were reduced to distribution graphs (unit hydrographs modified to show the proportional relation of their ordinates expressed as percentages of the total surface-runoff volume) to eliminate the natural differences in discharge rates arising from watershed size. Where several distribution graphs are available for a given watershed, a representative graph may be resolved by using recommended procedures (12, 13, and 22).

In this study, the method used was influenced by the inconsistency of the original data. The times-of-occurrence and magnitudes of the peak discharge rates were considered the most significant factors. When the individual graphs, plotted with a common time of beginning of surface runoff, showed small time variations at the peak discharge, an average graph was obtained by finding the
average peak stage and time and sketching a mean graph to conform to the individual graphs as closely as possible (12). If, on the other hand, the composite plot indicated extreme horizontal scattering so as to restrict the graphic determination of an average peak, the graphs were positioned to a location of best fit by giving preference to the following properties in decreasing order of importance; maximum ordinate, time of occurrence of precipitation excess,

ascending limb of the hydrograph, and descending limb of the hydrograph. The average period of rise and peak discharge were then obtained and a representative graph constructed by successive trial plottings.

The representative distribution graph of a watershed was designated the empirical graph. The term "empirical" was adopted to infer that the graph was developed from empirical data and to avoid the possibility of misinterpretation conveyed by the words mean or average.
The empirical graph for watershed 5 developed from 4 individual storms is shown in Fig. 2. It should be noted in Fig. 2 that the time scale has been transposed to coincide with zero discharge and zero time of the empirical graph. The discharge readings are plotted at the midpoints of the time interval that each represents. That is, \( Q_1 \), the percentage of total flow occurring during the first 5-min time increment, 0-5 min, is plotted at 2.5 min. Similarly, \( Q_2 \), the percentage of total flow occurring during the second 5-min time increment, 5 min to 10 min, is plotted at 7.5 min, and so on. The peak discharge was always computed irrespective of whether or not it fell at the midpoint of an interval. For this watershed, the agreement of the individual graphs is very good, and little difficulty arose in developing the empirical graph. For certain other watersheds, however, the separate graphs exhibited considerable scatter. For those cases, the development of the empirical graph was somewhat subjective, relative to locating the position of best fit.

Development of Dimensionless Graphs.—The empirical graphs for the 42 watersheds were further modified to a standardized form to avoid inconsistencies in the time-increments used in their description. Each graph was adjusted with its ordinate values expressed in percentage flow based on a time increment equal to 1/4 the period of rise (\( \% \text{ flow}/0.25 \text{ PR} \)) and the abscissa as the ratio of any time, \( t \), divided by the period of rise, \( \text{PR} \), the time from beginning of surface runoff to the occurrence of peak discharge (Fig. 3). The empirical graphs described in this manner were referred to as dimensionless graphs.

The time-increment duration of 0.25 \( \text{PR} \) was chosen for the following reasons: (1) The period of rise was ascertained to be an important time characteristic of a given watershed, (2) The use of 0.25 \( \text{PR} \) enables definition of the rising limb at four points and (3) The shape of the hydrograph was retained by using this sized increment.

Fitting of the Dimensionless Graph to the Two-Parameter Gamma Distribution.—The methods presently used to evaluate the parameters of Eqs. 1b and 2 from hydrologic data are somewhat limiting. Nash (15) suggests that the method of moments can be used to estimate the parameters, \( k \) and \( n \), provided both rainfall and runoff records are available. This procedure is, however, a cumbersome and laborious task. Edson (6) has given a nomograph for determining the parameters of Eq. 1a from the time of occurrence and magnitude of the peak discharge rate of a known unit graph. The obvious limitation in using the nomograph is that it does not utilize all the experimental data in the estimation but considers only the peak ordinate.

Efficient estimates of the unit-graph parameters (Eqs. 1b and 2) can be obtained if consideration is given to the analogy between the unit graph and the two-parameter or incomplete gamma distribution. The equation of the skew statistical frequency curve is given by the relationship

\[
f(x) = \frac{N(\gamma)q}{\Gamma(q)} e^{-\gamma x} x^{q-1} \quad \ldots \ldots \ldots \ldots \ldots (3)
\]

in which \( f(x) \) is any "ordinate" value; \( x \) is any "x" value; \( N \) represents total frequency or number of observations of \( x \); \( q \) and \( \gamma \) are shape and scale parameters, respectively; \( \Gamma \) denotes the gamma function; and \( e \) is the base of natural logarithms. Obviously, Eqs. 1b, 2, and 3 define the same curve when the following equalities exist, \( f(x) = Qt \); \( N = V \); \( x = t \); \( q = m = n \); and \( \gamma = y = 1/k \). Because of the similarity, it follows that the statistical procedures applied in
FIG. 3.—DIMENSIONLESS GRAPH AND FITTED TWO-PARAMETER GAMMA DISTRIBUTION FOR WATERSHED 5

FIG. 4.—THEORETICAL AND EXPERIMENTAL RELATIONSHIPS BETWEEN PARAMETERS \( q \) AND \( \gamma' \) OF DIMENSIONLESS GRAPHS
fitting the two-parameter gamma distribution to experimental data can be used to obtain estimators of \( q \) and \( \gamma \) for the unit hydrograph.

Because the dimensionless graph is simply a modified form of the unit graph, Eq. 3 can be used to describe its geometry after appropriate changes have been made to the constants. Unlike the unit graph, the value of \( N \) (Eq. 3) for the dimensionless graph remains constant, independent of watershed size. By simple computation, it can be shown that the geometry of the dimensionless graph can be approximated by the relationship (7)

\[
Q_t/P_R = \frac{25.0 (\gamma')^q}{\Gamma(q)} \left( e^{-\gamma't/P_R} \right) \left( \frac{t}{P_R} \right)^{q-1} \ldots (4)
\]

in which \( Q_t/P_R = \% \text{ flow}/0.25 P_R \) at any given \( t/P_R \) value; \( \gamma' \) = dimensionless parameter equal to the product, \( \gamma P_R \); and \( q, \Gamma \) and \( e \) = as described under Eq. 3. According to Eq. 4, with the values of \( q, \gamma' \), and \( P_R \) known, the dimensionless graph, distribution graph, and unit graph for the watershed can be developed.

The dimensionless graph of each watershed was processed through an IBM 650 computer to obtain the maximum likelihood estimators of the parameters, \( q \) and \( \gamma' \) (8). The experimental and fitted curves for watershed 5 are shown in Fig. 3.

A statistical, Chi-square test was applied to each fitted curve in an attempt to evaluate the goodness of fit. In no case was a significant Chi-square value \( (P \leq 0.05) \) obtained to reject the hypothesis that the fitted and actual curves are of the same population. However, in some cases, their agreement was sufficiently poor, particularly within the crest segment, to invalidate the use of the fitted curve from a hydrologic aspect. (A complete set of figures showing the fitted and actual curves for each watershed, complete with detailed discussions concerning problems involved in the fitting process and in evaluating the goodness of fit in terms of hydrologic acceptance, are given elsewhere (7).) As a result, an arbitrary "point" criterion was established; whereby, the values of \( q \) and \( \gamma' \) from the fitted curves that agreed within 20% of the dimensionless graph at the peak ordinate were used. This procedure reduced the number of watersheds in later investigations to 33.

Relation Between the Parameters, \( q \) and \( \gamma' \).—The parameters, \( q \) and \( \gamma' \), of Eq. 4 describing the dimensionless graph are linearly related. This relationship can be developed considering that at the peak, \( \frac{dQ_t}{P_R} = 0 \), \( t/P_R = 1 \), and \( Q_t/P_R \) is a maximum. By setting the first differential of Eq. 4 equal to zero and substituting \( t/P_R = 1 \) into the result, it follows that

\[
q = 1 + \gamma' \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5)
\]

Eq. 5 states that for the dimensionless graph the variables plot as a straight line with an intercept value and slope equal to unity.

As shown in Fig. 4, the experimental results deviate somewhat from the theoretical expression. The least squares line fitted to these data is defined by the regression

\[
q = 1.445 + 0.873 \gamma' \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6)
\]
The difference between the theoretical and experimental curves is indicative of the inability of the fitted graphs to achieve proper positioning of the peak ordinate. Although this difference was found to be statistically significant, it was not considered of sufficient magnitude to restrict the validity of the fitted curves, particularly if consideration is given to the subjectiveness involved in positioning the initial graphs and the scatter of the original data. In practical applications, Eq. 5 would always be used to obtain the peak at a value of \( \frac{t}{P_R} = 1 \).

**Evaluation of the Storage Factor, \( P_R/\gamma' \), from the Watershed Characteristic, \( L/\sqrt{SC} \).**—Use of the method of maximum likelihood as a fitting procedure provides that the variables, \( q, \gamma', \) and \( \frac{t}{P_R} \), are related in the following manner

\[
\frac{q}{\gamma'} = \frac{t}{P_R} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (7a)
\]

in which \( \frac{t}{P_R} \) is the mean value of the ratio for the dimensionless graph (19). By substituting \( \gamma = \gamma'/P_R \) into Eq. 7a, it follows that

\[
\frac{q}{\gamma} = \frac{t}{P_R} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (7b)
\]

in which \( \gamma \) is the scale parameter of the unit hydrograph (Eq. 3) and has the dimensions of the reciprocal of time.

In order to expedite the desired correlation, it is helpful to consider the relationship between the storage constant, \( k \) (Eq. 2) and the variables \( P_R \) and \( \gamma' \) for the instantaneous unit graph. From the equalities previously listed, it can be shown

\[
k = \frac{1}{\gamma} = \frac{1}{\gamma'} = \frac{P_R}{\gamma'} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (8)
\]

The parameter, \( P_R/\gamma' \), like the storage constant, \( k \), measures the storage properties of a watershed or the travel time required for water to pass through a given reach. It was, therefore, referred to as a storage factor.

The magnitude of the storage factor for a given watershed is relatively independent of the effects of rainfall duration. As a consequence, the differences in magnitudes of these factors for different watersheds can be attributed to differences in physical characteristics. R. K. Linsley, F. ASCE, (11) and C. O. Clark, M. ASCE, (4) have shown the storage constant, \( k \), to be related to the watershed characteristics, area, length and slope of the main stream. In addition, Linsley points out that the relationship may be influenced by regional differences.

From purely hydraulic considerations, the magnitude of the storage factor, \( P_R/\gamma' \), would be expected to vary directly with the length of the stream and inversely with some power of the channel slope. Thus, an attempt was made to relate \( P_R/\gamma' \) with the watershed parameter, \( L/\sqrt{SC} \), using data taken from 33 selected watersheds.

With the variables plotted on rectangular coordinate paper, it is evident that their relation is curvilinear. This property may possibly be attributed to use
of the square root of channel slope rather than some other power. J.C.I. Dooge, M. ASCE, (5) suggests that in loose boundary hydraulics, under given conditions of channel flow, travel time varies inversely with the cube root of channel slope. In view of these considerations, the functional form of the relation between the variables was assumed to be that of a power equation

\[ \frac{PR}{\gamma'} = a \left( \frac{L}{\sqrt{SC}} \right)^b \quad \ldots \ldots \ldots \ldots \quad (9) \]

in which \( a \) and \( b \), the coefficient and exponent, are evaluated from the experimental data.

Figs. 5, 6, and 7 show the storage factors, \( PR/\gamma' \), and watershed parameters, \( L/SC \), for the 33 watersheds plotted on logarithmic paper according to three regional groupings, Nebraska-Western Iowa, Central Iowa-Missouri-Illinois and Wisconsin, and Ohio. The regression equations for these data computed by the method of least squares are

Nebraska-Western Iowa. –

\[ \frac{PR}{\gamma'} = 7.40 \left( \frac{L}{\sqrt{SC}} \right)^{0.498} \quad \ldots \ldots \ldots \ldots \quad (10) \]

Central Iowa-Missouri-Illinois-Wisconsin. –

\[ \frac{PR}{\gamma'} = 9.27 \left( \frac{L}{\sqrt{SC}} \right)^{0.562} \quad \ldots \ldots \ldots \ldots \quad (11) \]

Ohio. –

\[ \frac{PR}{\gamma'} = 11.40 \left( \frac{L}{\sqrt{SC}} \right)^{0.531} \quad \ldots \ldots \ldots \ldots \quad (12) \]

in which \( PR/\gamma' \) is in minutes, \( L \) in miles and \( SC \) in per cent. For each group, the regression was found to be highly significant.

Usually the standard deviation from regression is used as a measure of the average deviation of the individual points from the regression line (17). However, when using logarithms, it is more convenient to use the coefficient of variation, \( CV \), the ratio of standard deviation from regression to the mean \( \bar{y} \) value, expressed in per cent, as a measure of this deviation (21). The values of \( CV \) for the three regression lines were computed as 28.0%, 30.7% and 29.1%, respectively. An additional index of the degree of association between the variables is given by the correlation coefficient, \( r \), listed on Figs. 5, 6, and 7.

An analysis of covariance of these data substantiated the selected grouping. This analysis showed the regression of \( PR/\gamma' \) values on \( L/\sqrt{SC} \) values for the Nebraska-Western Iowa watersheds to be significantly different from that for the Ohio watersheds. The experimental data follow parallel lines that pass through the mean logarithmic values of each group.

In contrast, the data from watersheds within Central Iowa, Illinois, Missouri, and Wisconsin adopt positions that correspond to each of the preceding two
FIG. 5.—RELATION OF STORAGE FACTOR, PR/\gamma', AND WATERSHED PARAMETER, L/\sqrt{S_C}, FOR WATERSHEDS IN NEBRASKA-WESTERN IOWA

FIG. 6.—RELATION OF STORAGE FACTOR, PR/\gamma', AND WATERSHED PARAMETER, L/\sqrt{S_C}, FOR WATERSHEDS IN CENTRAL IOWA-MISSOURI-ILLINOIS-WISCONSIN
FIG. 7.—RELATION OF STORAGE FACTOR, $P_R/\gamma'$, AND WATERSHED PARAMETER, $L/\sqrt{S_0}$, FOR WATERSHEDS IN OHIO

FIG. 8.—RELATION OF PARAMETER, $\gamma'$, AND PERIOD OF RISE, $P_R$, FOR 33 SELECTED WATERSHEDS
groups. From a statistical aspect, these could not be separated. However, because of the inability to associate the physical characteristics of the watersheds within these areas with either of the other regions, they were retained separately as an individual group shown in Fig. 6.

Comments.—From Figs. 5 and 7, it is evident that at a common value of \( \frac{L}{\sqrt{SC}} \), the storage factor, \( \frac{PR}{\gamma'} \), is higher for the Ohio watersheds than those for Nebraska-Western Iowa. This difference can be associated with differences in the geometry of the stream channels in the two regions. In Ohio, low flows are confined to shallow, vee-shaped channels that top to narrow, rounded valley bottoms. Even in the case of small flood waves, characteristic of those originating from a unit storm, overbank storage would probably be appreciable. In contrast, stream channels in the loessial area are in the form of deeply-entrenched, U-shaped gullies. For these areas, most flood flows would be confined within the channel. Also, within the range of watersheds studied, a given value of \( \frac{L}{\sqrt{SC}} \) is representative of a larger watershed in Ohio than in Nebraska-Western Iowa (7).

In applying the results, it is recommended that the empirical relation be selected from the group with comparable geologic, physiographic, and climatic conditions as the watershed in question. The 95-% confidence belts have been added to Figs. 5, 6, and 7 to facilitate the use of Eqs. 10, 11, and 12 as prediction equations.

Relation of Period of Rise, \( PR \), and Parameter, \( \gamma' \).—The results presented give a relationship between dimensionless-graph properties and a relationship between these properties and basin characteristics. They may be expressed in functional form as

\[
q = \phi(\gamma') \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (13)
\]

and

\[
\frac{PR}{\gamma'} = \phi' \left( \frac{L}{\sqrt{SC}} \right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (14)
\]

in which \( \phi \) and \( \phi' \) designate the function. With the value of \( \frac{L}{\sqrt{SC}} \) known, Eqs. 13 and 14 contain three unknowns. Thus, an additional expression is required before a solution can be obtained.

It was found that the variation in \( \gamma' \) could be significantly explained by linear regression with \( PR \) (Fig. 8). The equation of the line fitted to these data by the method of least squares is

\[
\gamma' = 2.876 + 0.0139 \cdot PR \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (15)
\]

For the regression the standard deviation from regression was computed as 1.253.

Because Eq. 15 is to be used in conjunction with Eqs. 10, 11, and 12, it is more convenient for computational purposes to express the result in the form

\[
\frac{PR}{\gamma'} = \frac{1}{\frac{2.876}{PR} + 0.0139} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (16)
\]
FIG. 9.—RELATION OF STORAGE FACTOR, $P_R/\gamma'$, AND PERIOD OF RISE, $P_R$

FIG. 10.—RELATION OF LAG, $t_L$, AND PERIOD OF RISE, $P_R$, FOR 94 SELECTED STORMS
Eq. 1b is shown plotted in Fig. 9 that can be used to solve for PR with PR/γ' known.

Comments.—The use of the results reported in Figs. 5, 6, 7, and 9 should be limited to watersheds having characteristics that fall within the limits of the experimental data. Additional study indicated that best reproducibility was obtained on the watersheds with L/√SC values less than 7 miles. It is evident from Eq. 1b that as PR approaches infinity, PR/γ' approaches a maximum value of 71.9 min. By using additional results given by Gray (7), it can be shown that this limit prohibits the use of the regressions to watersheds of the approximate sizes given in Table 2.

Selection of a Time Parameter.—Before applying the synthetic unit graph to a given storm sequence, it is necessary to have available a time parameter relating the salient features of rainfall and runoff for the area in question. Several forms of lag have been proposed for this purpose.

One of the more common forms of lag was introduced by F. F. Snyder, F. ASCE, (18) in 1938. He defined lag as the time difference between the center of mass of a surface-runoff-producing rain and the occurrence of peak discharge. For lag defined in this manner, in order to obtain a constant lag value for a given watershed, it is necessary to specify the storm type; otherwise,

<table>
<thead>
<tr>
<th>Region</th>
<th>Watershed area, in square miles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nebraska-Western Iowa</td>
<td>362</td>
</tr>
<tr>
<td>Central Iowa-Missouri-Illinois-Wisconsin</td>
<td>94</td>
</tr>
<tr>
<td>Ohio</td>
<td>82</td>
</tr>
</tbody>
</table>

due to the unsymmetrical nature of the hydrograph, the magnitude of lag of a given basin will vary with storm duration.

In this study lag, as defined by Snyder, was adopted. No restriction was placed on the storm type other than the unit-storm criteria, however. An initial attempt to determine the lag of each watershed proved unsuccessful because of the lack and inadequacy of precipitation data. As a result, an additional study was undertaken to find a more suitable time parameter, one which could be obtained for each watershed.

As shown by the Soil Conservation Service (20), lag, tL, and time of concentration of the watershed, TC, are related in the form \( t_L = 0.60 \, T_C \). For watersheds of the size used in the study, it is reasonable to assume PR approximates TC. It follows from there, that tL and PR would be related. These variables from 94 selected storms are shown plotted on logarithmic paper in Fig. 10. The regression line fitted to these data by the method of least squares is defined by the equation

\[ t_L = 0.996 \, PR^{1.005} \]  

(17)

For all practical cases, the values of the constant and exponent of Eq. 17 can be taken as unity, in which case \( t_L = PR \). That is, a given change in PR pro-
roduces an equal change in $t_L$. This simple linear regression between $t_L$ and $PR$ found in the data conforms to the form of the relationship suggested in previous comments.

A similar result was obtained by R. B. Hickok, F. ASCE, R. V. Keppel, and B. R. Rafferty (9) in their studies of rainfall and runoff records from 14 experimental watersheds in Arizona, New Mexico, and Colorado. They reported (9, p. 615)

"Rise time varied from 74 per cent to 145 per cent of the lag time (time from the center of mass of a limited block of intense rainfall to the re-

**TABLE 3.—COORDINATES OF THE SYNTHESIZED UNIT HYDROGRAPH**

<table>
<thead>
<tr>
<th>$\frac{t}{PR}$</th>
<th>Accumulated time, in min</th>
<th>$\frac{%}{0.25PR}$</th>
<th>Cumulative $%$ flow, 0.25$PR$</th>
<th>Unit graphs, in cfs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
</tr>
<tr>
<td>0.000</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.125</td>
<td>7.3</td>
<td>0.3</td>
<td>0.3</td>
<td>40</td>
</tr>
<tr>
<td>0.375</td>
<td>21.8</td>
<td>5.2</td>
<td>5.5</td>
<td>694</td>
</tr>
<tr>
<td>0.625</td>
<td>36.3</td>
<td>13.0</td>
<td>18.5</td>
<td>1,736</td>
</tr>
<tr>
<td>0.875</td>
<td>50.8</td>
<td>17.6</td>
<td>36.1</td>
<td>2,350</td>
</tr>
<tr>
<td>1.000</td>
<td>58.0</td>
<td>---</td>
<td>---</td>
<td>2,430</td>
</tr>
<tr>
<td>1.125</td>
<td>65.3</td>
<td>17.7</td>
<td>53.8</td>
<td>2,363</td>
</tr>
<tr>
<td>1.375</td>
<td>79.8</td>
<td>14.9</td>
<td>68.7</td>
<td>1,989</td>
</tr>
<tr>
<td>1.625</td>
<td>94.3</td>
<td>11.2</td>
<td>79.9</td>
<td>1,495</td>
</tr>
<tr>
<td>1.875</td>
<td>108.0</td>
<td>7.7</td>
<td>87.6</td>
<td>1,028</td>
</tr>
<tr>
<td>2.125</td>
<td>123.3</td>
<td>5.0</td>
<td>92.6</td>
<td>668</td>
</tr>
<tr>
<td>2.375</td>
<td>137.8</td>
<td>3.1</td>
<td>95.7</td>
<td>414</td>
</tr>
<tr>
<td>2.625</td>
<td>152.3</td>
<td>1.9</td>
<td>97.6</td>
<td>254</td>
</tr>
<tr>
<td>2.875</td>
<td>166.8</td>
<td>1.1</td>
<td>98.7</td>
<td>147</td>
</tr>
<tr>
<td>3.125</td>
<td>181.3</td>
<td>0.6</td>
<td>99.3</td>
<td>80</td>
</tr>
<tr>
<td>3.375</td>
<td>195.8</td>
<td>0.3</td>
<td>99.6</td>
<td>40</td>
</tr>
<tr>
<td>3.625</td>
<td>210.3</td>
<td>0.2</td>
<td>99.8</td>
<td>27</td>
</tr>
<tr>
<td>3.875</td>
<td>224.8</td>
<td>0.1</td>
<td>99.9</td>
<td>14</td>
</tr>
<tr>
<td>4.125</td>
<td>239.3</td>
<td>0.1</td>
<td>100.0</td>
<td>13</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100.0</strong></td>
<td></td>
<td><strong>13,352</strong></td>
<td></td>
</tr>
</tbody>
</table>

a Rounded to nearest 0.10%.

b Peak discharge rate; not included in total.

The association between the lag time used in the preceding and lag as used herein is assumed. For short-duration storms, as used in the development of unit graphs for small watersheds, the center of a limited block of intense rain and the mass center of the surface-runoff-producing rain would be nearly coincident. The variances from regression of the $t_L$ values and $PR$ values were approximately equal. The coefficients of variation were computed to be 27.1% and 25.7%, respectively.

On the basis of this evidence, it was concluded that the period of rise, $PR$, of the hydrograph could be used as an effective time parameter to relate the
salient features of rainfall and runoff on a given watershed. The result is generally applicable only for uniformly-distributed, short-duration, high-intensity storms occurring over small watershed areas.

**Application of Results.**

Problem.—Define the unit hydrograph for a watershed, 5 sq miles in area, that falls within a region of comparable geologic, physiographic, and climatic

conditions as those of Western Iowa. The following information was obtained from an available topographic map: \( L = 3.80 \) miles and \( SC = 0.57\% \).

**Procedures.**

Step 1. Determine parameters; \( PR, \gamma' \) and \( q \).

A. With \( L/\sqrt{SC} = 3.80/\sqrt{0.57} = 5.03 \) miles, enter Fig. 5 and select; \( PR/\gamma' = 16.6 \) min.

B. With \( PR/\gamma' = 16.6 \) min, enter Fig. 9 and obtain; \( PR = 58 \) min. Therefore, \( \gamma' = 58/16.6 = 3.494 \)

C. Set the peak to fall at \( t/PR = 1 \), by substituting \( \gamma' = 3.494 \) into Eq. 5 and solve for \( q = 4.494 \).

Step 2. Compute the ordinates of the dimensionless graph.

A. Using Eq. 4, compute the % flow/0.25 \( PR \) at the respective values of \( t/PR \) - 0.125, 0.375, 0.625... and every succeeding increment of \( t/PR \) = 0.250, until the sum of the ordinates approximates 100% (Table 3). Also
compute the peak percentage. At the peak,

\[ Q(1) = \frac{25.0 \times (3.494)^{4.494}}{\Gamma(4.494)} e^{-3.494(1)} (1)^{4.494} = 18.2\% \]

Step 3. Develop the unit hydrograph.

A. Compute the necessary conversion factor.

(a) Volume of unit hydrograph, V

\[ V = 1 \text{ in.} \times 5 \text{ mile}^2 \times 640 \text{ acre per mile}^2 \times \frac{1}{12 \text{ in. per ft}} \times 43560 \text{ ft}^2 \text{ per acre} = 11,616,000 \text{ ft}^3 \]

(b) Volume of dimensionless graph, \( V_D \)

\[ V_D = \Sigma \text{ cfs} \times 0.25 \times 58 \text{ min} \times 60 \text{ sec per min} = 870 \Sigma \text{ cfs - sec.} \]

Because the two volumes, V and \( V_D \), must be equal, it follows that \( \Sigma \text{ cfs} = 11,616,000/870 = 13,352 \text{ cfs} \).

B. Convert the dimensionless graph ordinates to cfs.

\[ \frac{Q_t}{PR} = \left( \frac{\% \text{ flow}/0.25 \text{ PR}}{100} \right) \Sigma \text{ cfs} \]

Therefore, at the peak,

\[ Q_p = 18.2/100 \times 13,352 = 2,430 \text{ cfs.} \]

C. Convert the time base of the dimensionless graph to absolute time units. At the peak, \( t/PR = 1 \); therefore, \( t = 58 \text{ min.} \)

Step 4. Plot the unit hydrograph (Fig. 11).

According to Fig. 10, the time of beginning of surface, runoff should be placed coincident with the centroid of precipitation. For convenience of computation, the unit hydrograph should be associated with unit-storm periods of 0.25 PR duration.

CONCLUSIONS

A workable synthetic procedure for the development of the unit hydrograph for small watersheds from measurable physical characteristics is presented. Hydrologic and topographic data from 42 small watersheds in the states of Illinois, Iowa, Missouri, Ohio, and Wisconsin were analyzed. For each watershed, a representative distribution graph, the so-called empirical graph, was derived and modified to a dimensionless form based on the period of rise, PR, as the time parameter.

The two-parameter gamma distribution described by the parameters, \( q \) and \( \gamma' \), was fitted to each dimensionless graph and the maximum likelihood estimators of the parameters obtained. Relationships were established so that the parameters, PR, q and \( \gamma' \) could be evaluated from the topographic characteristics L and SC of a given basin. With PR, q and \( \gamma' \) known, the dimensionless graph, distribution graph, and unit hydrograph for the basin can be described.

The following conclusions were derived from this study.

1. In general, the two-parameter gamma distribution can be used to describe the dimensionless graph, distribution graph, or unit hydrograph.
2. The storage factor, $P_{R/\gamma'}$, can be predicted with reasonable success from the watershed factor, $L/\sqrt{SC}$, provided consideration is given to regional influence.

3. The parameter, $\gamma'$, of the two-parameter gamma distribution describing the dimensionless graph can be estimated from the period of rise.

4. For a given watershed, the dimensionless graph, distribution graph, and unit hydrograph can be derived from the watershed characteristic, $L/\sqrt{SC}$;

ACKNOWLEDGMENTS

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APPENDIX.—SELECTED BIBLIOGRAPHY


5. "Synthetic Unit Hydrographs Based on Triangular Inflow," by J. C. I. Dooge, thesis presented to the State Univ. of Iowa in Iowa City, Iowa, in 1956, in partial fulfillment of the requirements for the degree of Master of Science.


