RATE OF ADVANCE OF LAMINAR OVERLAND FLOW

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SUMMARY

Experiments were conducted on a model flume in study of the rate of advance of the liquid front for the case of laminar, overland flow. The experimental values were compared with those obtained from solutions of different hydrodynamic and kinematic equations defining the phenomenon.

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INTRODUCTION

The importance of the advance problem to the design and operation of surface irrigation systems has long been recognized. During the last fifty years, numerous analyses and studies have been made of this phenomenon. Among the most notable of these works include those by; Crevat (1907), Friedrich (1912), Lewis and Milne (1938), Criddle et al. (1956), Kostiakov (1960), Ostromeski (1960), Philip and Farrell (1964), Kruger and Bassett (1965), Olsen (1965), Wilke and Smerdon (1965, 1967), and Hart et al. (1968). In general, the approaches used to study the problem fall into one of two categories, namely: hydrodynamics or kinematics. In all studies of the advance of overland flow, either the differential or the integral equations are derived and solved.

However, few of these solutions have been substantiated by experimental data. Accurate measurements of the overland flow phenomenon in the field are not only difficult to obtain but also represent

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space-time integrated values of many factors affecting the system. Similarly, obtaining of necessary experimental data from "controlled" laboratory experiments has been inhibited due to the difficulty in constructing a "physically-scaled" model on which pertinent variables can be controlled and the time variation in the infiltration process synthesized.

The primary objective of the study presented in the paper was to test the applicability of different hydrodynamic and kinematic equations for describing the advance phenomenon. Solutions of several theoretical equations were compared with experimental data obtained from a laminar flow model having a time-decreasing infiltration rate.

**HYDRODYNAMIC EQUATIONS**

**Dynamic Equation** - By assuming that;

1. The slope of the porous bed is small,
2. The velocity components of rainfall and the fluid particles which infiltrate the porous bed are zero in the direction of stream flow,
3. The rainfall intensity and impact have a negligible effect on the velocity and momentum distribution of the fluid, and,
4. The air resistance and surface tension forces are sufficiently small to be neglected;
Lin (1970) has shown that the dynamic equation defining unsteady flow in a rectangular flume can be derived from the momentum principle as

\[(1-\beta) \frac{V}{gy} \frac{\partial V}{\partial t} + \frac{1}{g} \frac{\partial V}{\partial t} + \frac{\partial y}{\partial x} + \beta \frac{V}{g} \frac{\partial V}{\partial x} + \frac{V^2}{g} \frac{\partial \beta}{\partial x} = S_0 - S_f - \beta \frac{V(R-I)}{gy} \ldots (1)\]

where

- $\beta$ = momentum coefficient,
- $V$ = average velocity of the flow,
- $g$ = acceleration due to gravity,
- $y$ = depth of the flow,
- $t$ = time,
- $x$ = distance of flow cross-section,
- $S_0$ = slope of the porous bed,
- $S_f$ = friction slope,
- $R$ = rainfall intensity, a function of $x$ and $t$, and
- $I$ = infiltration rate, a function of $x$ and $t$.

**Continuity Equation** - The continuity equation for unsteady, spatially-varied flow in a rectangular channel is

\[\frac{\partial y}{\partial t} + y \frac{\partial V}{\partial x} + V \frac{\partial y}{\partial x} = R - I \ldots (2)\]

**Generalized Hydrodynamic Equation** - Combining Equations 1 and 2, and simplifying, one obtains a generalized equation defining overland flow for variable rainfall and infiltration rates as
\[
\frac{\partial y}{\partial x} = \frac{S_o - S_f - \frac{1}{g} \left( \frac{V}{y} [2\beta(R-I)] + (1-2\beta) \frac{\partial v}{\partial t} + V^2 \frac{\partial \beta}{\partial x} \right)}{1 - F^2} \tag{3}
\]

where \( F \) is the Froude number, which is defined, when the slope of the porous bed is small, as \( F = V/\sqrt{gy/\beta} \).

Numerical solutions of Equations 1, 2 and 3, derived by Lin (1970), can be used to solve the laminar overland flow problem over a porous bed. In later parts of this paper, these solutions are referred to as the "Hydrodynamic Solutions".

**KINEMATIC EQUATIONS**

**Conservation Equation** - The cumulative depth of infiltration and the distance of advance of the flow tip can usually be expressed as power law functions of time as

\[
z = k\tau^\alpha \tag{4}
\]

\[
X = h\tau^\beta \tag{5}
\]

where

\( z \) = cumulative depth of infiltration,

\( \tau \) = infiltration opportunity time, \((0 \leq \tau \leq t)\)

\( X \) = distance of advance of the flow tip at time, \( t \),

\( k, h \) = coefficients, and,

\( \alpha, \beta \) = power constants \((0 \leq \alpha \leq 1, 0 \leq \beta \leq 1)\).
Kiefer (1959), and Hart et al. (1968) expressed the total volume of water infiltrated into a unit width of soil as (see Fig. 1).

\[
\bar{V} = \lambda X Z
\]  

(6)

where \( \bar{V} \) = the total volume of water infiltrated,

\( Z \) = the cumulative depth of infiltration at the upstream end of the dyke when the flow tip reaches a distance of advance, \( X \), and

\( \lambda \) = a constant.

The constant, \( \lambda \), is geometrically equal to the ratio of the average depth of infiltration over the length of advance to the depth of infiltration at the upstream end of the strip. Mathematically, it may be calculated as

\[
\lambda = \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)}
\]  

(7)

where \( \Gamma \) = gamma function. Values of \( \lambda \) for different values of \( \alpha \) and \( \beta \) are plotted in Fig. 2. As shown in the figure, \( \lambda \) may vary in the range: \( 1 > \lambda > \frac{1}{2}, \ (0 \leq \alpha \leq 1, \ 0 \leq \beta \leq 1) \).

Using the conservation equation, the length of advance \( X \) may therefore be calculated as

\[
X = \frac{qt}{c + \lambda z}
\]  

(8)
FIG. 1  Schematic Sketch of Overland Flow Phenomenon
Figure 2 - $\lambda$ AS A FUNCTION OF $\alpha$ AND $\beta$
where \( c = \) average surface depth; a function of time, \( t, \) and
\( q = \) constant discharge (input) rate per unit width
applied to the strip.

**Philip and Farrell Solution** - Assuming the average surface depth
of the flow remains constant with time, Lewis and Milne (1938) derived
the following integral equation for the advance of overland flow down
a border strip under a constant input rate as:

\[
q_t = c^*X + \int_0^t z(t - \xi) X'(\xi) \, d\xi \quad \ldots (9)
\]

where \( c^* = \) average surface depth, taken as a constant,
\( \xi = \) time variable \((0 \leq \xi \leq t),\)
\( z(t-\xi) = \) a cumulative depth of infiltration; a function
of infiltration opportunity time, \((t-\xi),\) and
\( X'(\xi) = \left(\frac{dx}{dt}\right)_t = \xi.\)

Philip and Farrell (1964) applied the Laplace transformation to
obtain an analytical solution to the integral equation for different
types of infiltration equations. When the infiltration equation takes
the form of Equation 4; the solution reduces to the dimensionless form,

\[
\frac{c^*X}{q_t} = \sum_{n=0}^{\infty} \frac{\left(\frac{-kt}{c^*}\right)^n}{\Gamma(1 + \alpha)} \frac{\Gamma(1 + \alpha)^n}{\Gamma(2 + n \alpha)} \quad \ldots (10)
\]

Wilke and Smerdon (1965) used Equation 10 to obtain a family of
solutions for five values of \( kt^{\alpha}/c^* \) ranging from 1.25 to 1.75 and for
different values of $\alpha$.

A regression equation for $\alpha = \frac{1}{2}$, passing through the point

$\left( \frac{k t^{\alpha}}{c^*} = 0, \frac{q t}{c^*} = 1 \right)$ for values of $\frac{k t^{\alpha}}{c^*} < 2$ was

$$\frac{q t}{c^*} = 1 + 0.7165 \left( \frac{k t^{\alpha}}{c^*} \right)$$

... (11)

EXPERIMENTAL WORK

A rectangular transparent plastic channel, 7$\frac{1}{2}$ feet long and 2$\frac{1}{2}$ inches wide, (see Fig. 3) was used for the experiments to determine the advance and flow profiles.

Infiltration on the model was simulated with horizontally-placed capillary tubes. The flow rate in these tubes can be defined by the equation, (Lin and Gray, 1971),

$$M = \frac{\pi d^2}{4} \sqrt{\frac{g d^2}{16 \mu} (H + h_c)} \, t^\frac{3}{2}$$

... (12)

where

$M$ = volumetric flow into the capillary tube after time, $t$,

$d$ = diameter of the tube,

$\rho$ = density of the fluid,

$\mu$ = dynamic viscosity of the fluid,

$H$ = pressure head at the entrance to the tube, and

$h_c$ = capillary head at the tip of flow.
FIG. 3 Experimental Apparatus
Three tests of laminar overland flow were conducted on the model using ink-dyed glycerine (approximately 93% concentration) as the fluid. In each trial the temperature of the fluid was measured and in most cases found to be within the range from 76°F to 80°F. Within this temperature range both the density and surface tension of the glycerine remain reasonably constant, however, its viscosity changes appreciably. Thus, the viscosity of the glycerine was measured at the beginning of each test.

Measurements were made of (a) the advance of the flow tip, (b) the depth of the flow profile, and (c) the infiltration length and pattern, with time. The flow profile was obtained from readings taken from 21 self-generating photocells placed at 4-inch spacings attached to the external wall of the flume and activated by light from a fluorescent lamp placed on the opposite wall. The entire set of photocells was scanned in 5 seconds. Output was recorded on a continuous strip chart recorder. Measurements of the infiltration pattern were obtained from vertical photographs taken of the advance in the capillary tubes. The photocell readings and photographs were used to determine the advance of flow.

The major differences between the tests are summarized in Table 1.
TABLE 1 - Comparison of Tests

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Total Discharge (cm³/sec)</th>
<th>Channel Slope</th>
<th>Fluid Viscosity (poise)</th>
<th>Capillary Tube Diameter (cm)</th>
<th>Fluid Density (gm³/cm³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.882</td>
<td>0.0000</td>
<td>2.85</td>
<td>0.2446</td>
<td>1.246</td>
</tr>
<tr>
<td>2</td>
<td>6.277</td>
<td>0.0043</td>
<td>2.57</td>
<td>0.2446</td>
<td>1.246</td>
</tr>
<tr>
<td>3</td>
<td>6.477</td>
<td>0.0087</td>
<td>2.51</td>
<td>0.2446</td>
<td>1.246</td>
</tr>
</tbody>
</table>

RESULTS AND DISCUSSIONS

The experimental data of distance of advance versus time are plotted on logarithmic paper in Fig. 4. From the figure, it can be observed that the data plot as straight lines and can be expressed by the power law function of time (Equation 5) with a value of $\beta = 0.80$.

Measurements of the average surface depth, c, and the cumulative infiltration at the upstream end, Z, at different times obtained from the three tests are plotted in Figs. 5 and 6, respectively. From Fig. 5, it is evident that the average surface depth; which generally is assumed to be constant (e.g. Friedrich (1912), Lewis and Milne (1938), Philip and Farrell (1964), Wilke and Smerdon (1965)), increases non-linearly with time. The rate of increase is most rapid during the initial period following introduction of the fluid into the flume.

The experimental data presented in Table 1 and Figs. 4, 5 and 6 were used to compare the validity of the different equations: (a)
Kinematic Conservation Equation (Equation 8), (b) Philip and Farrell Solution (Equation 11) and (c) Hydrodynamic Solution (Lin, 1970), to express the advance of laminar overland flow. In applying these equations to the problem it should be noted that

1. The value of \( \lambda \) (Equation 7) was calculated using values of \( \alpha = 0.5 \) (Equation 12) and \( \beta = 0.80 \) (Fig. 4).

2. The average surface depth of flow used in the Philip and Farrell solution of the advance problem was taken as

\[
c^* = \frac{1}{T} \int_{0}^{T} c \, dt
\]

where \( T \) = time interval under consideration
(240 sec in this study).

Inasmuch as the values of \( kt^\alpha/c^* \ (\alpha = \frac{1}{2}) \) for the three tests were less than 1, the regression equation of the Philip and Farrell proposed by Wilke and Smerdon (Equation 11) was used to calculate the advance.

Values of the advance of the liquid front calculated by each of the methods and the experimental results are plotted in Figs. 7-9. From the results presented, it is evident that the calculated values obtained with the "Kinematic Conservation Equation" and the "Hydrodynamic Equation" are in close agreement with the experimental values obtained from the model. Conversely, however, there are large
Figure 4: ADVANCE CURVES FOR LAMINAR OVERLAND FLOW MODEL

$\beta = 0.80$ FOR $X = h t^\beta$

$\square$, ○, △ EXPERIMENTAL DATA

TEST - 1  TEST - 2  TEST - 3
discrepancies between the curves calculated by the "Philip and Farrell" solution and the data. The major differences exist at short times after introducing the fluid to the flume; during which the lengths-of-advance calculated by the theoretical solution are less than the measured values, and at longer times where the "calculated advance" is greater than the "measured". These results suggest that Equation 13 tends to over-estimate the average surface depth, c*, at short times after commencing a test and to under-estimate c* during the later stages. Also, the agreement between calculated and experimental data depends on the general applicability of the regression equation (Equation 11) in defining the laminar, overland flow phenomenon.

The preceding results should not be construed as invalidating the application of the "Philip-Farrell" solution to determine advance in field irrigation systems at times when the average surface depth along the irrigation strip does not change appreciably with time - a condition assumed in the development of the equation. They do, however, substantiate that the success of the solution in predicting advance is limited to this condition.

SUMMARY AND CONCLUSIONS

Several tests were conducted on a model flume on which the phenomenon of laminar, overland flow over a porous bed could be synthesized. Values of the advance of the liquid front obtained from the flume were compared with solutions of different hydrodynamic and kinematic equations used to define the overland flow phenomenon.
Figure 8 ADVANCE CURVES FOR LAMINAR OVERLAND FLOW - TEST 2
Figure 9 ADVANCE CURVES FOR LAMINAR OVERLAND FLOW - TEST 3
From the results obtained it was found that,

(1) The "length-of-advance" of laminar, overland flow can be approximated as a power law function of time.

(2) The average surface depth of laminar, overland flow increases non-linearly with time. This property limits the use of solutions such as that proposed by Philip and Farrell to predict advance on models – where short advance times are involved.

(3) The advance of the flow tip calculated by the "Kinematic Conservation Equation" and the numerical solutions of "Hydrodynamic Equations" developed by Lin (1970) were in close agreement with values obtained from the model.
ACKNOWLEDGEMENT

The authors gratefully acknowledge the financial assistance provided by the National Research Council of Canada in support of this study.
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